MA11210 Differential Equations – Exercise Sheet 3 – Solutions

1a$*$) The general solution of $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 12y = 24x^2$ is of the form

$$y(x) = \text{Complementary Function} + \text{Particular Integral}$$

To find the Complementary Function, y_c, we first solve the auxiliary equation:

$$m^2 + m - 12 = 0 \quad \Rightarrow \quad m_1 = -4, \; m_2 = 3.$$

As the roots are real and distinct, the Complementary Function is:

$$y_c = Ae^{-4x} + Be^{3x}.$$

where A and B are arbitrary constants.

Since the forcing term is a monomial of degree 2, we try the following trial function in order to determine the Particular Integral:

$$y_p(x) = a + bx + cx^2, \text{ so that } \frac{dy_p}{dx} = b + 2cx \text{ and } \frac{d^2y_p}{dx^2} = 2c.$$

Substituting into the differential equation gives:

$$\begin{aligned} 2c + (b + 2cx) - 12(a + bx + cx^2) &= 24x^2 \\ \Rightarrow -12a + b + 2c + (-12b + 2c)x - 12cx^2 &= 24x^2 \end{aligned}$$

Comparing coefficients of:

$$\begin{array}{rlcl} x^2) & -12c = 24 & \Rightarrow & c = -2 \\ x) & 2c - 12b = 0 & \Rightarrow & b = -\frac{1}{3} \\ 1) & -12a + b + 2c = 0 & \Rightarrow & a = -\frac{13}{36} \end{array}$$

Therefore, a Particular Integral is $-\frac{13}{36} - \frac{1}{3}x - 2x^2$ and the general solution is

$$\boxed{y(x) = Ae^{-4x} + Be^{3x} - \frac{13}{36} - \frac{1}{3}x - 2x^2}$$

[6]

1b) The general solution of $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 2e^{-x}$ is of the form

$$y(x) = \text{Complementary Function} + \text{Particular Integral}$$

To find the Complementary Function, y_c, we first solve the auxiliary equation:

$$m^2 + 6m + 9 = 0 \Rightarrow m_1 = m_2 = -3$$

As there is one repeated real root, the Complementary Function is:

$$y_c = (A + Bx)e^{-3x}$$

where A and B are arbitrary constants.

Since the forcing term is an exponential function, we try the following trial function in order to determine the Particular Integral:

$$y_p(x) = ae^{-x}, \text{ so that } \frac{dy_p}{dx} = -ae^{-x} \text{ and } \frac{d^2y_p}{dx^2} = ae^{-x}.$$

Substituting into the differential equation gives:

$$\begin{aligned} ae^{-x} - 6ae^{-x} + 9ae^{-x} &= 2e^{-x} \\ \Rightarrow 4ae^{-x} = 2e^{-x} \end{aligned}$$

Therefore, $a = \frac{1}{2}$ and a Particular Integral is $\frac{1}{2}e^{-x}$. Hence, the general solution is

$$\boxed{y(x) = (A + Bx)e^{-3x} + \frac{1}{2}e^{-x}}$$

1c∗) The general solution of $\frac{d^2y}{dx^2} + 4y = 10\sin(3x)$ is of the form

$$y(x) = \text{Complementary Function} + \text{Particular Integral}$$

To find the Complementary Function, y_c, we first solve the auxiliary equation:

$$m^2 + 4 = 0 \quad \Rightarrow \quad m = \pm 2i.$$

As the roots are complex conjugates (in fact purely imaginary), the Complementary Function is:

$$y_c = A\sin(2x) + B\cos(2x),$$

where A and B are arbitrary constants.

Seek the particular integral in the form

$$y_p(x) = a\sin(3x) + b\cos(3x).$$

Substitution into the ODE yields

$$-5a\sin(3x) - 5b\cos(3x) = 10\sin(3x),$$

whence $a = -2$ and $b = 0$.
Combining the above,

$$y = A\sin(2x) + B\cos(2x) - 2\sin(3x).$$

[5]

1d) The general solution of $\frac{d^2y}{dx^2} + 9y = 9\sin 3x$ is of the form

$$y(x) = \text{Complementary Function} + \text{Particular Integral}$$

To find the Complementary Function, y_c we first solve the auxiliary equation:

$$m^2 + 9 = 0 \quad \Rightarrow \quad m_1 = -3i,\ m_2 = 3i$$

As the roots are complex conjugate, the Complementary Function is:

$$y_c = A\cos 3x + B\sin 3x$$

where A and B are arbitrary constants.

Since the usual trial function associated with the forcing term (namely $a\cos 3x + b\sin 3x$) can be derived from the Complementary Function (by setting $A = a$ and $B = b$), we try the following trial function in order to determine the Particular Integral: $y_p(x) = x(a\cos 3x + b\sin 3x)$, so that

$$\frac{dy_p}{dx} = (a + 3bx)\cos 3x + (b - 3ax)\sin 3x \text{ and } \frac{d^2y_p}{dx^2} = (6b - 9ax)\cos 3x - (6a + 9bx)\sin 3x.$$

Substituting into the differential equation gives:

$$\begin{aligned}(6b - 9ax)\cos 3x - (6a + 9bx)\sin 3x + 9(ax\cos 3x + bx\sin 3x) &= 9\sin 3x \\ \Rightarrow 6b\cos 3x - 6a\sin 3x &= 9\sin 3x\end{aligned}$$

Comparing coefficients of:

$$\begin{array}{lll} \cos 3x\,) & 6b = 0 & \Rightarrow \quad b = 0 \\ \sin 3x\,) & -6a = 9 & \Rightarrow \quad a = -\frac{3}{2} \end{array}$$

Therefore, a Particular Integral is $-\frac{3}{2}x\cos 3x$ and the general solution is

$$\boxed{y(x) = A\cos 3x + B\sin 3x - \frac{3}{2}x\cos 3x}$$

1e$*$) The general solution of $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 10e^{3x}$ is of the form

$$y(x) = \text{Complementary Function} + \text{Particular Integral.}$$

To find the Complementary Function, y_c, we first solve the auxiliary equation:

$$m^2 - 6m + 9 = 0 \Rightarrow m_1 = m_2 = 3.$$

As there is one repeated real root, the Complementary Function is:

$$y_c = (A + Bx)e^{3x}$$

where A and B are arbitrary constants. [3]

Since the usual trial function associated with the forcing term (namely ae^{3x}) can be derived from the Complementary Function (by putting $A = a$ and $B = 0$) (NB: xe^x can also be derived from the Complementary Function), we try the following trial function in order to determine the Particular Integral: $y_p(x) = ax^2e^{3x}$, so that

$$\frac{dy_p}{dx} = ae^{3x}(3x^2 + 2x) \text{ and } \frac{d^2y_p}{dx^2} = ae^{3x}(9x^2 + 12x + 2).$$

Substituting into the differential equation gives:

$$\begin{aligned} ae^{3x}(9x^2 + 12x + 2 - 18x^2 - 12x + 9x^2) &= 10e^{3x} \\ \Rightarrow 2ae^{3x} &= 10e^{3x} \end{aligned}$$

Therefore, $a = 5$ and a Particular Integral is $5x^2e^{3x}$. Hence, the general solution is

$$\boxed{y(x) = (A + Bx)e^{3x} + 5x^2e^{3x}}$$

[5]

2a) First, we find the general solution of $y'' + 3y' + 2y = 30\cos t$, which is of the form

$$y(t) = \text{Complementary Function} + \text{Particular Integral}$$

To find the Complementary Function, y_c, we first solve the auxiliary equation:

$$m^2 + 3m + 2 = 0 \quad \Rightarrow \quad m_1 = -2,\ m_2 = -1$$

As the roots are real and distinct, the Complementary Function is:

$$y_c = Ae^{-2t} + Be^{-t}$$

where A and B are arbitrary constants.

Since the forcing term is a multiple of a trigonometric function, we try the following trial function in order to determine the Particular Integral: $y_p(t) = a\cos t + b\sin t$, so that

$$y_p'(t) = -a\sin t + b\cos t \text{ and } y_p''(t) = -a\cos t - b\sin t.$$

for some constants a and b to be determined. Substituting into the differential equation gives:

$$\begin{aligned} -a\cos t - b\sin t + 3(-a\sin t + b\cos t) + 2(a\cos t + b\sin t) &= 30\cos t \\ \Rightarrow (a+3b)\cos t + (b-3a)\sin t &= 30\cos t \end{aligned}$$

Comparing coefficients of:

$$\begin{aligned} \cos t)\quad & a + 3b = 30 \\ \sin t)\quad & b - 3a = 0 \end{aligned}$$

Solving the above simultaneous equations gives, $a = 3$ and $b = 9$. Therefore, a Particular Integral is $\cos t + 3\sin t$ and the general solution is

$$\boxed{y(t) = Ae^{-2t} + Be^{-t} + 3\cos t + 9\sin t}$$

In order to apply the initial condition $y'(0) = -1$, we first differentiate the general solution to obtain:

$$y'(t) = -2Ae^{-2t} - Be^{-t} - 3\sin t + 9\cos t.$$

Applying the initial conditions:

$$\begin{aligned} 4 &= y(0) = A + B + 3 \Rightarrow A + B = 1 \\ -1 &= y'(0) = -2A - B + 9 \Rightarrow 2A + B = 10 \end{aligned}$$

Solving the above simultaneous equations gives $A = 9$ and $B = -8$. Hence the particular solution is

$$\boxed{y(t) = 9e^{-2t} - 8e^{-t} + 3\cos t + 9\sin t}$$

2b$*$) First, we find the general solution of $y'' - 2y' + 2y = 2t + 5e^{-t}$, which is of the form

$$y(t) = \text{Complementary Function} + \text{Particular Integral}$$

To find the Complementary Function, y_c, we first solve the auxiliary equation:

$$m^2 - 2m + 2 = 0 \Rightarrow m_1 = 1 - i,\ m_2 = 1 + i$$

As the roots are complex conjugate, the Complementary Function is:

$$y_c = e^t(A\cos t + B\sin t)$$

where A and B are arbitrary constants.

Since the forcing term is a sum of a monomial of degree 1 and multiple of an exponential function, we try the following trial function in order to determine the Particular Integral:

$$y_p(t) = a + bt + ce^{-t}, \text{ so that } y_p'(t) = b - ce^{-t} \text{ and } y_p''(t) = ce^{-t}.$$

for some constants a and b to be determined. Substituting into the differential equation gives:

$$\begin{aligned} ce^{-t} - 2(b - ce^{-t}) + 2(a + bt + ce^{-t}) &= 2t + 5e^{-t} \\ \Rightarrow 2a - 2b + 2bt + 5ce^{-t} = 2t + 5e^{-t} \end{aligned}$$

Comparing coefficients of:

$$\begin{array}{rlcl} e^{-t}) & 5c = 5 & \Rightarrow & c = 1 \\ t) & 2b = 2 & \Rightarrow & b = 1 \\ 1) & 2a - 2b = 0 & \Rightarrow & a = 1 \end{array}$$

Therefore, a Particular Integral is $1 + t + e^{-t}$ and the general solution is

$$\boxed{y(t) = e^t(A\cos t + B\sin t) + 1 + t + e^{-t}}$$

In order to apply the initial condition $y'(0) = -1$, we first differentiate the general solution to obtain:

$$y'(t) = e^t((A + B)\cos t + (B - A)\sin t) + 1 - e^{-t}.$$

Applying the initial conditions:

$$\begin{aligned} 0 &= y(0) = A + 2 \Rightarrow A = -2 \\ -1 &= y'(0) = A + B \Rightarrow B = 1 \end{aligned}$$

Hence the particular solution is

$$\boxed{y(t) = e^t(\sin t - 2\cos t) + 1 + t + e^{-t}}$$

[6]

2c) First, we find the general solution of $y'' + 4y = 10e^t - 4\cos 2t$, which is of the form

$$y(t) = \text{Complementary Function} + \text{Particular Integral}$$

To find the Complementary Function, y_c, we first solve the auxiliary equation:

$$m^2 + 4 = 0 \Rightarrow m_1 = -2i,\ m_2 = 2i$$

As the roots are complex conjugate, the Complementary Function is:

$$y_c = A\cos 2t + B\sin 2t$$

where A and B are arbitrary constants.

Since the usual trial function associated with a term of the forcing term (namely $a\cos 2t + b\sin 2t$) can be derived from the Complementary Function, we try the following trial function in order to determine the Particular Integral:

$$y_p(t) = at\cos 2t + bt\sin 2t + ce^t, \text{ so that}$$

$y_p'(t) = (a+2bt)\cos 2t + (b-2at)\sin 2t + ce^t$ and $y_p''(t) = (4b-4at)\cos 2t - (4a+4bt)\sin 2t + ce^t$.

for some constants a and b to be determined. Substituting into the differential equation gives:

$$\begin{aligned}(4b-4at)\cos 2t - (4a+4bt)\sin 2t + ce^t + 4(at\cos 2t + bt\sin 2t + ce^t) &= 10e^t - 4\cos 2t \\ \Rightarrow 4b\cos 2t - 4a\sin 2t + 5ce^t = 10e^t - 4\cos 2t\end{aligned}$$

Comparing coefficients of:

$$\begin{array}{rlcl} \sin 2t\,) & -4a = 0 & \Rightarrow & a = 0 \\ \cos 2t\,) & 4b = -4 & \Rightarrow & b = -1 \\ e^t\,) & 5c = 10 & \Rightarrow & c = 2 \end{array}$$

Therefore, a Particular Integral is $-t\sin 2t + 2e^t$ and the general solution is

$$\boxed{y(t) = A\cos 2t + B\sin 2t - t\sin 2t + 2e^t}$$

In order to apply the initial condition $y'(0) = 6$, we first differentiate the general solution to obtain:

$$y'(t) = (2B - 2t)\cos 2t - (2A + 1)\sin 2t + 2e^t.$$

Applying the initial conditions:

$$\begin{aligned} 5 &= y(0) = A + 2 \Rightarrow A = 3 \\ 6 &= y'(0) = 2B + 2 \Rightarrow B = 2 \end{aligned}$$

Hence the particular solution is

$$\boxed{y(t) = 3\cos 2t + 2\sin 2t - t\sin 2t + 2e^t}$$

2d$*$) First, we find the general solution of $y'' + 4y' + 4y = 10 + 8t - 6te^{-2t}$, which is of the form

$$y(t) = \text{Complementary Function} + \text{Particular Integral}$$

To find the Complementary Function, y_c, we first solve the auxiliary equation:

$$m^2 + 4m + 4 = 0 \Rightarrow m_1 = m_2 = -2$$

As there is one repeated real root, the Complementary Function is:

$$y_c = (A + Bt)e^{-2t}$$

where A and B are arbitrary constants.

Since the usual trial function associated with a term of the forcing term (namely $(c+dt)e^{-2t}$) can be derived from the Complementary Function (by setting $A = c$ and $B = d$), we try the following trial function in order to determine the Particular Integral:

$$y_n(t) = a + bt + t^2(c+dt)e^{-2t}, \text{ so that } y_n'(t) = b + (2ct + (3d-2c)t^2 - 2dt^3)e^{-2t},$$

$$y_n''(t) = (2c + (6d-8c)t + (4c-12d)t^2 + 4dt^3)e^{-2t}$$

for some constants a, b, c and d to be determined. Substituting into the differential equation gives:

$$\begin{aligned}(2c + (6d-8c)t + (4c-12d)t^2 + 4dt^3)e^{-2t} + 4(b + (2ct + (3d-2c)t^2 - 2dt^3)e^{-2t})\\ + 4(a + bt + t^2(c+dt)e^{-2t}) = 10 + 8t - 6te^{-2t}\\ \Rightarrow 4a + 4b + 4bt + (2c + 6dt)e^{-2t} = 10 + 8t - 6te^{-2t}\end{aligned}$$

Comparing coefficients of:

$$\begin{array}{rlcl} e^{-2t}\,) & 2c = 0 & \Rightarrow & c = 0 \\ te^{-2t}\,) & 6d = -6 & \Rightarrow & d = -1 \\ t\,) & 4b = 8 & \Rightarrow & b = 2 \\ 1\,) & 4a + 4b = 10 & \Rightarrow & a = \frac{1}{2} \end{array}$$

Therefore, a Particular Integral is $\frac{1}{2} + 2t - t^3e^{-2t}$ and the general solution is

$$\boxed{y(t) = (A + Bt - t^3)e^{-2t} + \frac{1}{2} + 2t}$$

In order to apply the initial condition $y'(0) = 1$, we first differentiate the general solution to obtain:

$$y'(t) = (B - 2A - 2Bt - 3t^2 + 2t^3)e^{-2t} + 2.$$

Applying the initial conditions:

$$\begin{aligned} 2 &= y(0) = A + \frac{1}{2} \Rightarrow A = \frac{3}{2} \\ 1 &= y'(0) = B - 2A + 2 \Rightarrow B = 2 \end{aligned}$$

Hence the particular solution is

$$\boxed{y(t) = \left(\frac{3}{2} + 2t - t^3\right)e^{-2t} + \frac{1}{2} + 2t}$$

[8]

3) The general solution of $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 10\sin 3x - 5e^{-x}$ is of the form

$$y(x) = \text{Complementary Function} + \text{Particular Integral}$$

To find the Complementary Function, y_c, we first solve the auxiliary equation:

$$m^3 + m^2 + 4m + 4 = 0 \Rightarrow m^2(m+1) + 4(m+1) = 0 \Rightarrow (m+1)(m^2+4) = 0$$

$$\Rightarrow m_1 = -1,\ m_2 = -2i,\ m_3 = 2i$$

As there is one real root and a pair of complex conjugate roots, the Complementary Function is:

$$y_c = Ae^{-x} + B\cos 2x + C\sin 2x$$

where A, B and C are arbitrary constants.

Since the usual trial function (ae^{-x}) corresponding to a term of the forcing term ($5e^{-x}$) can be derived from the Complementary Function (by setting $A = a$), we try the following trial function in order to determine the Particular Integral:

$$\begin{aligned} y_p(x) &= axe^{-x} + b\cos 3x + c\sin 3x, \text{ so that} \\ y_p'(x) &= ae^{-x}(-x+1) - 3b\sin 3x + 3c\cos 3x, \\ y_p''(x) &= ae^{-x}(x-2) - 9b\cos 3x - 9c\sin 3x \\ y_p'''(x) &= ae^{-x}(-x+3) + 27b\sin 3x - 27c\cos 3x \end{aligned}$$

Substituting into the differential equation gives:

$$\begin{aligned} ae^{-x}(-x+3) + 27b\sin 3x - 27c\cos 3x + ae^{-x}(x-2) - 9b\cos 3x - 9c\sin 3x \\ +4(ae^{-x}(-x+1) - 3b\sin 3x + 3c\cos 3x) + 4(axe^{-x} + b\cos 3x + c\sin 3x) &= 10\sin 3x - 5e^{-x} \\ \Rightarrow 5ae^{-x} - (15c + 5b)\cos 3x + (15b - 5c)\sin 3x &= 10\sin 3x - 5e^{-x} \end{aligned}$$

Comparing coefficients of:

$$\begin{array}{rlcl} e^{-x}\,) & 5a = -5 & \Rightarrow & a = -1 \\ \cos 3x\,) & -15c - 5b = 0 & \Rightarrow & b = -3c \\ \sin 3x\,) & 15b - 5c = 10 & \Rightarrow & c = -\frac{1}{5} \Rightarrow b = \frac{3}{5} \end{array}$$

Therefore, a Particular Integral is $-xe^{-x} + \frac{3}{5}\cos 3x - \frac{1}{5}\sin 3x$ and the general solution is

$$\boxed{y(x) = (A - x)e^{-x} + B\cos 2x + C\sin 2x + \frac{3}{5}\cos 3x - \frac{1}{5}\sin 3x}$$

4$*$) First, we find the general solution of $y^{(iv)} - 4y'' = 8e^{2x} + 24x$, which is of the form

$$y(x) = \text{Complementary Function} + \text{Particular Integral}$$

To find the Complementary Function, y_c, we first solve the auxiliary equation:

$$m^4 - 4m^2 = 0 \Rightarrow m^2(m^2 - 4) = 0 \Rightarrow m_1 = m_2 = 0,\ m_3 = -2,\ m_4 = 2$$

As there is one root that is repeated twice and a pair of distinct real roots, the Complementary Function is:

$$y_c = A + Bx + Ce^{2x} + De^{-2x}$$

where A, B, C and D are arbitrary constants.

Since usual trial function $(ae^{2x} + b + cx)$ corresponding to the forcing term can be derived from the Complementary Function (by setting $C = a, A = b$ and $B = c$), we try the following trial function in order to determine the Particular Integral:

$$\begin{aligned} y_p(x) &= axe^{2x} + x^2(b + cx), \\ y_p'(x) &= ae^{2x}(2x + 1) + 2bx + 3cx^2, \\ y_p''(x) &= ae^{2x}(4x + 4) + 2b + 6cx \\ y_p'''(x) &= 4ae^{2x}(2x + 3) + 6c \\ y_p^{(iv)}(x) &= 16ae^{2x}(x + 2) \end{aligned}$$

Substituting into the differential equation gives:

$$\begin{aligned} 16ae^{2x}(x + 2) - 4(ae^{2x}(4x + 4) + 2b + 6cx) &= 8e^{2x} + 24x \\ \Rightarrow 16ae^{2x} - 8b - 24cx &= 8e^{2x} + 24x \end{aligned}$$

Comparing coefficients of:

$$\begin{array}{rlcl} e^{2x}\,) & 16a = 8 & \Rightarrow & a = \frac{1}{2} \\ 1\,) & -8b = 0 & \Rightarrow & b = 0 \\ x\,) & -24c = 24 & \Rightarrow & c = -1 \end{array}$$

Therefore, a Particular Integral is $\frac{1}{2}xe^{2x} - x^3$ and the general solution is

$$\boxed{y(x) = A + Bx + Ce^{2x} + De^{-2x} + \frac{1}{2}xe^{2x} - x^3}$$

In order to apply the initial conditions, we first differentiate the general solution to obtain:

$$\begin{aligned} y'(x) &= B + 2Ce^{2x} - 2De^{-2x} + \frac{1}{2}e^{2x}(2x + 1) - 3x^2 \\ y''(x) &= 4Ce^{2x} + 4De^{-2x} + 2e^{2x}(x + 1) - 6x \\ y'''(x) &= 8Ce^{2x} - 8De^{-2x} + 2e^{2x}(2x + 3) - 6 \end{aligned}$$

Applying the initial conditions:

$$\begin{aligned} 3 &= y(0) = A + C + D \\ 1 &= y'(0) = B + 2C - 2D + \frac{1}{2} \\ 2 &= y''(0) = 4C + 4D + 2 \\ -2 &= y'''(0) = 8C - 8D + 6 - 6 \end{aligned}$$

Solving the above system of equations for A, B, C and D gives $A = 3,\ B = 1,\ C = -\frac{1}{8},\ D = \frac{1}{8}$. Hence the solution is

$$\boxed{y(x) = 3 + x - x^3 + \frac{1}{8}(4x - 1)e^{2x} + \frac{1}{8}e^{-2x}}$$

[7]