

MA11210 Differential Equations – Exercise Sheet 2 – Solutions

1. (a) Use an integrating factor (because the ODE is linear and first order), OR separable variables.
- (b) Solve by the method of separable variables (because $\frac{dy}{dx} = f(x)g(y)$, where $f(x) = \frac{1}{x-3}$ and $g(y) = y+2$), OR use an integrating factor (since it's also first order linear).
- (c) Solve by direct integration (the right hand side is a function only of the independent variable x).
- (d) ODE is homogeneous - making the substitution $y = vx$ will reduce the problem to separable form.
- (e) Use an integrating factor (because the ODE is linear and first order)

2a) We must find the solution of $xy^3 \frac{dy}{dx} = x^4 + y^4$ that satisfies the boundary condition $y(1) = 2$.

Rearranging the differential equation gives

$$\frac{dy}{dx} = \frac{x^4 + y^4}{xy^3}$$

This is a first order homogeneous differential equation, so we let $y = vx$, which implies that $\frac{dy}{dx} = x \frac{dv}{dx} + v$. Substituting these into the differential equation gives

$$\begin{aligned} x \frac{dv}{dx} + v &= \frac{x^4(1+v^4)}{x^4v^3} = \frac{1+v^4}{v^3} \\ \therefore x \frac{dv}{dx} &= \frac{1+v^4}{v^3} - v \\ &= \frac{1}{v^3} \end{aligned}$$

Separating the variables and integrating gives

$$\begin{aligned} \int v^3 dv &= \int \frac{1}{x} dx + A \quad (A \text{ arbitrary constant}) \\ \therefore \frac{1}{4}v^4 &= \ln|x| + A \end{aligned}$$

Substituting $v = \frac{y}{x}$ and rearranging gives the general solution:

$$y^4 = 4x^4(\ln|x| + A).$$

Applying the boundary condition $y(1) = 2$ gives $16 = 4A \Rightarrow A = 4$. Therefore the particular solution is

$$y^4 = 4x^4(\ln x + 4).$$

2b) We must find the solution of $x(2y + x)\frac{dy}{dx} = 3y^2 + 2xy$ that satisfies the boundary condition $y(2) = 4$.

Rearranging the differential equation gives

$$\frac{dy}{dx} = \frac{3y^2 + 2xy}{x(2y + x)}$$

This is a first order homogeneous differential equation, so we let $y = vx$, which implies that $\frac{dy}{dx} = x\frac{dv}{dx} + v$. Substituting these into the differential equation gives

$$\begin{aligned} x\frac{dv}{dx} + v &= \frac{x^2(3v^2 + 2v)}{x^2(2v + 1)} \\ \therefore x\frac{dv}{dx} &= \frac{3v^2 + 2v}{2v + 1} - v \\ &= \frac{v^2 + v}{2v + 1} \end{aligned}$$

Separating the variables and integrating gives

$$\begin{aligned} \int \frac{2v + 1}{v^2 + v} dv &= \int \frac{1}{x} dx + A \quad (A \text{ arbitrary constant}) \\ \therefore \ln |v^2 + v| &= \ln B|x| \quad (A = \ln B) \end{aligned}$$

Substituting $v = \frac{y}{x}$ and rearranging gives the general solution:

$$|y^2 + xy| = B|x|^3.$$

Applying the boundary condition $y(2) = 4$ gives $16 + 8 = 8B \Rightarrow B = 3$. Therefore the particular solution is

$$y^2 + xy = 3x^3.$$

3a) Let $x = u + x_0$ and $y = v + y_0$ where u and v are functions of x and x_0, y_0 are constants to be determined. Then $\frac{dv}{du} = \frac{dv}{dx} \frac{dx}{du} = \frac{dy}{dx}(1) = \frac{dy}{dx}$, so that:

$$\begin{aligned} \frac{dv}{du} &= \frac{a(u + x_0) + b(v + y_0) + c}{l(u + x_0) + m(v + y_0) + n} \\ &= \frac{au + bv + (ax_0 + by_0 + c)}{lu + mv + (lx_0 + my_0 + n)} \end{aligned}$$

This differential equation is homogeneous if:

$$ax_0 + by_0 + c = 0 \tag{1}$$

$$lx_0 + my_0 + n = 0 \tag{2}$$

Replacing equation (2) with $a \times (2) - l \times (1)$ gives the equivalent system,

$$ax_0 + by_0 = -c \tag{3}$$

$$(am - bl)y_0 = cl - an \tag{4}$$

The above system of equations has a solution since it is given that $am - bl \neq 0$. Therefore, x_0 and y_0 can be found so that the original differential equation is reduced to the homogeneous equation:

$$\frac{dv}{du} = \frac{au + bv}{lu + mv}.$$

3b) Let $u = lx + my + n$ then

$$\begin{aligned} \frac{du}{dx} &= l + m \frac{dy}{dx} = l + \frac{m(ax + by + c)}{lx + my + n} \\ &= l + \frac{amx + bmy + cm}{u} = l + \frac{blx + bmy + cm}{u} \quad (\text{since } am = bl) \\ &= l + \frac{b(lx + my) + cm}{u} = l + \frac{b(u - n) + cm}{u} \end{aligned}$$

Therefore, we have reduced the original equation to the variable separable differential equation

$$\frac{du}{dx} = F(u)$$

where $F(u) = \frac{(b+l)u+cm-bn}{u}$.

3ci) We must find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{x + y + 2}{x - y + 2}.$$

Note that it is of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{lx + my + n}$$

where $am - bl = -2 \neq 0$. Therefore, by 2a) we can reduce the differential equation to a first order homogeneous differential equation by letting $x = u + x_0$ and $y = v + y_0$ for some appropriate constants x_0 and y_0 . Now

$$\frac{dy}{dx} = \frac{dv}{du} = \frac{u + v + x_0 + y_0 + 2}{u - v + x_0 - y_0 + 2}.$$

Choose x_0 and y_0 so that they satisfy

$$\begin{aligned} x_0 + y_0 + 2 &= 0 \\ x_0 - y_0 + 2 &= 0 \end{aligned}$$

That is, choose $x_0 = -2$ and $y_0 = 0$. Then

$$\frac{dv}{du} = \frac{u + v}{u - v}$$

This is a first order homogeneous differential equation, so we let $v = wu$, which implies that $\frac{dv}{du} = u \frac{dw}{du} + w$. Substituting these into the differential equation gives

$$\begin{aligned} u \frac{dw}{du} + w &= \frac{u(1 + w)}{u(1 - w)} \\ \therefore u \frac{dw}{du} &= \frac{1 + w}{1 - w} - w \\ &= \frac{1 + w^2}{1 - w} \end{aligned}$$

Separating the variables and integrating gives

$$\int \frac{1-w}{1+w^2} dw = \int \frac{1}{u} du + A \quad (A \text{ arbitrary constant})$$
$$\therefore \int \frac{1}{1+w^2} dw - \frac{1}{2} \int \frac{2w}{1+w^2} dw = \ln B|u| \quad (A = \ln B)$$
$$\therefore \tan^{-1} w - \frac{1}{2} \ln(1+w^2) = \ln B|u|$$

Substituting $w = \frac{v}{u}$ and rearranging gives

$$\tan^{-1} \left(\frac{v}{u} \right) = \ln \left[B(v^2 + u^2)^{\frac{1}{2}} \right].$$

Substituting $v = y$ and $u = x + 2$ gives the general solution:

$$\tan^{-1} \left(\frac{y}{x+2} \right) = \ln \left[B(y^2 + (x+2)^2)^{\frac{1}{2}} \right].$$

3cii) First we note that the differential equation is of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{lx + my + n}$$

where $a = b = -2, c = 6, l = m = 2, n = 5$. Furthermore, we see that $am = 4 = bl$. Therefore, from 2b) we see that by letting $u = 2x + 2y + 5$ we can reduce the differential equation to a variable separable differential equation:

$$\frac{du}{dx} = \frac{(b+l)u + cm - bn}{u} = \frac{22}{u}$$

Separating the variables and integrating gives:

$$\int u du = \int 22 dx + A \quad (A \text{ arbitrary constant})$$
$$\Rightarrow \frac{1}{2} u^2 = 22x + A$$
$$\Rightarrow (2x + 2y + 5)^2 = 44x + 2A \quad (\text{since } u = 2x + 2y + 5)$$
$$\Rightarrow x^2 + y^2 + 2xy - 6x + 5y = B \quad \text{where } B \text{ is some arbitrary constant.}$$

4a) We must find the general solution of the first order, linear differential equation

$$\frac{dy}{dx} + 3y = 6.$$

The integrating factor is $e^{\int 3 dx} = e^{3x}$. Multiplying the equation by the integrating factor gives

$$\frac{d}{dx} \{ e^{3x} y \} = 6e^{3x}.$$

Integrating gives

$$e^{3x} y = 2e^{3x} + A \quad (A \text{ arbitrary constant}).$$

Therefore, the general solution is

$$y = 2 + Ae^{-3x}.$$

4b) We must find the general solution of the first order, linear differential equation

$$\frac{dx}{dt} - \frac{2x}{t} = 2t^3 + 4t^2e^{4t}.$$

The integrating factor is $e^{\int -\frac{2}{t} dt} = \frac{1}{t^2}$. Multiplying the equation by the integrating factor gives

$$\frac{d}{dt} \left\{ \frac{1}{t^2} x \right\} = 2t + 4e^{4t}.$$

Integrating gives

$$\frac{x}{t^2} = t^2 + e^{4t} + A \quad (A \text{ arbitrary constant}).$$

Therefore, the general solution is

$$x = t^4 + (e^{4t} + A)t^2.$$

4c) We must first find the general solution of the first order, linear differential equation $\frac{dy}{dx} + y = e^{-x}$, and then apply the boundary condition $y(0) = 1$. We multiply both sides of the ODE by an integrating factor $e^{\int 1 dx} = e^x$, giving

$$\frac{d}{dx} \{e^x y\} = 1.$$

Thus $e^x y = x + A$ (A arbitrary constant), and so $y = xe^{-x} + Ae^{-x}$. We now apply the boundary condition to find the hitherto arbitrary constant A :

$$0e^0 + Ae^0 = A = 1.$$

Thus the particular solution is given by

$$y = (x + 1)e^{-x}.$$

4d) We must first find the general solution of the first order, linear differential equation $x \frac{dy}{dx} + 2y = 10x^2$, and then apply the boundary condition $y(1) = 3$. We first divide throughout by x in order to get the equation into a standard form:

$$\frac{dy}{dx} + \frac{2}{x}y = 10x.$$

We then multiply throughout by an integrating factor $e^{\int \frac{2}{x} dx} = x^2$, yielding

$$x^2 \frac{dy}{dx} + 2xy = 10x^3,$$

which we can write as

$$\frac{d}{dx} \{x^2y\} = 10x^3.$$

Integrating with respect to x gives $x^2y = \frac{5}{2}x^4 + A$ where A is an arbitrary constant, which upon rearranging gives the explicit solution

$$y = \frac{5}{2}x^2 + Ax^{-2}.$$

We now apply the boundary condition that when $x = 1$, $y = 3$. Thus

$$3 = \frac{5}{2} + A,$$

so $A = \frac{1}{2}$. It follows that the particular solution is given by

$$y = \frac{1}{2}(5x^2 + x^{-2}).$$