MA11210 Differential Equations – Exercise Sheet 1 – Solutions

1.(i) $$\frac{dy}{dx} = \sin 2x - e^{3x}$$

$$\begin{aligned} y &= \int \sin 2x dx - \int e^{3x} dx + A \quad \text{(where } A \text{ is an arbitrary constant)} \\ \therefore \quad y &= -\frac{1}{2}\cos 2x - \frac{1}{3}e^{3x} + A \quad \text{is the general solution.} \end{aligned}$$

1.(ii) $$\frac{dy}{dx} = \cot 4x$$

$$\begin{aligned} y &= \int \cot 4x \, dx + A = \int \frac{\cos 4x}{\sin 4x} \, dx + A \quad \text{(where } A \text{ is an arbitrary constant)} \\ \therefore \quad y &= = \frac{1}{4} \ln|\sin 4x| + A \qquad \text{(since } \int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)|) \end{aligned}$$

is the general solution.

1.(iii) $$\frac{dy}{dx} = \frac{5}{(x+3)(x-2)}$$

$$\begin{aligned} \text{Now,} \quad \frac{5}{(x+3)(x-2)} &= \frac{A}{x+3} + \frac{B}{x-2} \quad \text{for some constants } A, B \\ &= \frac{A(x-2) + B(x+3)}{(x+3)(x-2)} \\ \text{Equating the numerators gives} \quad 5 &= A(x-2) + B(x+3) \end{aligned}$$

Put $x = 2$: $5 = 5B \quad \Rightarrow B = 1.$
Put $x = -3$: $5 = -5A \Rightarrow A = -1.$

$$\begin{aligned} \therefore \quad y &= \int \frac{5}{(x+3)(x-2)} dx + C \quad \text{(where } C \text{ is an arbitrary constant)} \\ &= \int \frac{1}{x-2} dx - \int \frac{1}{x+3} dx + C \\ &= \ln|x-2| - \ln|x+3| + \ln D \quad (C = \ln D) \\ \therefore \quad y &= \ln\left(D\left|\frac{x-2}{x+3}\right|\right) \quad \text{is the general solution.} \end{aligned}$$

1.(iv) $$\frac{dy}{dx} = \frac{x+2}{x(x+1)^2}$$

$$\begin{aligned} \text{Now,} \quad \frac{x+2}{x(x+1)^2} &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \quad \text{for some constants } A, B, C \\ &= \frac{A(x+1)^2 + Bx(x+1) + Cx}{x(x+1)^2} \\ \text{Equating the numerators gives} \quad x+2 &= A(x+1)^2 + Bx(x+1) + Cx \end{aligned}$$

Put $x = -1$: $\quad 1 = -C \Rightarrow C = -1$.
Put $x = 0$: $\quad 2 = A \quad \Rightarrow A = 2$.
Compare coefficients of $x^2 : 0 = A + B \Rightarrow B = -2$.

$$\begin{aligned} \therefore \quad y &= \int \frac{2}{x} dx - \int \frac{2}{x+1} dx - \int \frac{1}{(x+1)^2} dx + D \quad \text{(where } D \text{ is an arbitrary constant)} \\ &= 2\ln|x| - 2\ln|x+1| + \frac{1}{x+1} + D \\ \therefore \quad y &= \ln\left(\frac{x}{x+1}\right)^2 + \frac{1}{x+1} + D \quad \text{is the general solution.} \end{aligned}$$

1.(v) $\quad \dfrac{dy}{dx} = \dfrac{2}{9\cos^2 2x - \sin^2 2x}$

Let $t = \tan 2x$ so that $\dfrac{dt}{dx} = 2\sec^2 2x$. Then

$$\begin{aligned} y &= \int \frac{2}{9\cos^2 2x - \sin^2 2x} dx \\ &= \int \frac{2}{(9 - \tan^2 2x)\cos^2(2x)} dx \\ &= \int \frac{2\sec^2 2x}{9 - \tan^2 2x} dx \\ &= \int \frac{1}{9 - t^2} \frac{dt}{dx} dx \\ &= \int \frac{1}{9 - t^2} dt \\ \text{Now,} \quad \frac{1}{9 - t^2} &= \frac{A}{3 - t} + \frac{B}{3 + t} \quad \text{for some constants } A,\ B \\ &= \frac{A(3 + t) + B(3 - t)}{9 - t^2} \\ \text{Equating the numerators gives} \quad 1 &= A(3 + t) + B(3 - t) \end{aligned}$$

Put $t = 3$: $\quad 1 = 6A \Rightarrow 1 = 6A$
Put $t = -3$: $\quad 1 = 6B \Rightarrow 1 = 6B$

$$\begin{aligned} \text{Therefore,} \quad y &= \frac{1}{6}\int \frac{1}{3 - t} dt + \frac{1}{6}\int \frac{1}{3 + t} dt + C \quad \text{where } C \text{ is an arbitrary constant} \\ &= -\frac{1}{6}\ln|3 - t| + \frac{1}{6}\ln|3 + t| + \ln D \quad (C = \ln D) \\ \therefore \quad y &= \ln\left\{D\left|\frac{3 + \tan 2x}{3 - \tan 2x}\right|^{\frac{1}{6}}\right\} \quad \text{is the general solution.} \end{aligned}$$

1.(vi) $\quad \dfrac{dy}{dx} = x\ln(x+1)^2 \quad (x > -1)$

Therefore, $\dfrac{dy}{dx} = 2x\ln(x+1)$, since $x+1>0$.

Integrating by parts gives
$$\begin{aligned} y &= (\ln(x+1))(x^2) - \int (x^2)\left(\frac{1}{x+1}\right) dx + A \quad (A \text{ arbitrary constant}) \\ &= x^2\ln(x+1) - \int \left\{x - 1 + \frac{1}{x+1}\right\} dx + A \quad \text{by partial fractions} \\ &= x^2\ln(x+1) - \frac{1}{2}x^2 + x - \ln(x+1) + A \\ \therefore \quad y &= (x^2-1)\ln(x+1) - \frac{1}{2}x^2 + x + A \quad \text{is the general solution.} \end{aligned}$$

2. (i) First order, first degree, linear.
(ii) Second order, fourth degree, non-linear.
(iii) Third order, first degree, linear.
(iv) Second order, first degree, non-linear.
(v) Second order, first degree, linear.
(vi) Fifth order, fourth degree, non-linear.
(vii) First order, first degree, linear.
(viii) First order, first degree, non-linear.

3.(i) $\sin x \dfrac{dy}{dx} = y\cos x + \cos x$, and we are given that $y(\frac{3\pi}{2}) = -2$.

Rearranging gives $\sin x \frac{dy}{dx} = (y+1)\cos x$. Separating the variables and integrating gives

$$\begin{aligned} \int \frac{1}{y+1}\, dy &= \int \cot x\, dx + A \quad (A \text{ arbitrary constant}) \\ \ln|y+1| &= \ln|\sin x| + A \\ \text{Therefore, } y &= B\sin x - 1, \text{ where } B \text{ is an arbitrary constant, is the general solution.} \end{aligned}$$

Applying the boundary condition gives:

$$-2 = y\left(\frac{3\pi}{2}\right) = B\sin\frac{3\pi}{2} - 1 = -B - 1 \Rightarrow B = 1$$

Therefore, $y = \sin x - 1$ is the particular solution.

3.(ii) $x^2(y^2+1)\dfrac{dy}{dx} - y^2 = 0$, and we are given that $y(2) = 1$.

Rearranging gives $\dfrac{y^2+1}{y^2}\dfrac{dy}{dx} = \dfrac{1}{x^2}$. Integrating gives:

$$\begin{aligned} \int \left\{1 + \frac{1}{y^2}\right\} dy &= \int \frac{1}{x^2}\, dx + A \quad \text{where } A \text{ is an arbitrary constant} \\ y - \frac{1}{y} &= -\frac{1}{x} + A \\ \therefore \quad y - \frac{1}{y} + \frac{1}{x} &= A \quad \text{is the general solution.} \end{aligned}$$

Applying the boundary condition $y(2) = 1$ gives:

$$A = 1 - \frac{1}{1} + \frac{1}{2} = \frac{1}{2}.$$

Therefore, the particular solution is $y - \frac{1}{y} + \frac{1}{x} = \frac{1}{2}$.

3.(iii) $$(2x+3)\frac{dy}{dx} = \frac{1}{(y+2)(y-3)}$$ and we are given that $y(-2) = 1$.

Rearranging gives, $(y+2)(y-3)\dfrac{dy}{dx} = \dfrac{1}{2x+3}$. Integrating gives:

$$\begin{aligned}
\int (y+2)(y-3)\, dy &= \int \frac{1}{2x+3}\, dx + A \quad \text{where } A \text{ is an arbitrary constant} \\
\int \left\{y^2 - y - 6\right\}\, dy &= \frac{1}{2}\ln|2x+3| + A \\
\therefore \quad \frac{1}{3}y^3 - \frac{1}{2}y^2 - 6y &= \frac{1}{2}\ln|2x+3| + A \quad \text{is the general solution.}
\end{aligned}$$

Applying the boundary condition $y(-2) = 1$ gives:

$$\frac{1}{3} - \frac{1}{2} - 6 = \frac{1}{2}\ln|-1| + A \implies A = -\frac{37}{6}.$$

Therefore, the particular solution is given by

$$\frac{1}{3}y^3 - \frac{1}{2}y^2 - 6y = \frac{1}{2}\ln|2x+3| - \frac{37}{6}.$$