MA11210 Differential Equations – Assignment 4

A mathematical murder mystery

Any resemblance to real persons or timetables, living or dead (including their murderous tendencies), is purely coincidental.

1. Newton's Law of Cooling states that the rate of change of temperature of a cooling body is directly proportional to the difference in temperature between the body and its surroundings.

   At ten minutes to six on a cold Thursday evening in March, police arrive on the scene of a gruesome murder in the Physical Sciences Library, where the top-of-the-range heating system keeps the temperature at a steady 20°C. The victim's body has suffered severe head trauma, seemingly having been struck by a length of lead piping that is found nearby. At precisely 18:00, the deceased's temperature is measured and found to be 25.0°C. Half an hour later it was re-measured and had decreased to 23.3°C. The on-the-scene reporter from the Cambrian News comments that this is much lower than usual body temperature (37.0°C) and concludes the body 'must have been lying there for quite a while', although a hard-working first year student who left the library shortly before 15:00 reports the victim was alive in the library at that time.

   After a lengthy interview procedure during which Dr Pitchford reports that the probability of him being the murderer was zero (and proving this to be the case via Kolmogrorov's axioms of probability), police narrow down their list of suspects to three, all of whom have gaps in their alibis:

   - Dr Vellender was lecturing Differential Equations (MA11210) from 16:00-17:00, and Asymptotic Methods in Mechanics (MA34210) from 17:00-18:00, but has no alibi before 16:00.
   - Prof. Cox was lecturing Mathematical Models of Biological Systems (MA34920) non-stop from 15:00-17:00, allowing no break in the middle. His whereabouts after 17:00 are unclear.
   - Dr Douglas was lecturing Lebesgue Integration (MA37010) from 15:00-16:00 and was in his weekly Zumba class from 17:00-18:00. In the intervening hour, he was not seen.

   (a) Write down a differential equation for the temperature T (a function of time t) of the body, along with an initial condition.

   (b) Use an appropriate method to find the explicit particular solution of this differential equation.

   (c) Find the likely time of death and thus discover the identity of the murderous lecturer.

2. An oil droplet of mass 0.2 g falls vertically under gravity from rest in air. The drag on the droplet due to air resistance is 1.6×10^{-3} Newtons when its speed is 0.4 m/s. Assuming that the magnitude of the drag force is proportional to the speed of the droplet and that the gravitational acceleration is 9.8m/s^2 find:

   (a) the velocity at time t seconds;

   (b) the distance traveled at time t seconds; and

   (c) the limiting velocity.

3. Let Q be the amount of carbon-14 present in a substance at time t and let Q_0 be the original amount. Assume that the amount of carbon-14 disintegrates at a rate proportional to the amount of carbon-14 currently present, and that the half-life of carbon-14 is 5,730 years.

   (a) Derive an expression for Q as a function of time t, by solving an appropriate differential equation that models the disintegration of carbon-14.[Show all workings to gain full credit.]

   (b) Certain remains are discovered in which the current residual amount of carbon-14 is 20% of the original amount. Determine the age of these remains.

4. The population of locusts in a certain area increases at a rate proportional to the current population size. In the absence of other factors the population doubles each week. Initially, the area contains 500,000 locusts. Predators such as birds and rodents eat 50,000 locusts per day.

   (a) Determine the population of locusts at any time.

   (b) Does the population of locusts eventually die out? If so, determine the time it takes.

   (c) What is the minimum number of locusts that would need to be eaten per day in order that the population eventually dies out.

5. A reasonable model for the growth of a population that is constrained by limited resources is

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{K}\right)$$

   for some positive constants k and K, where p is the size of the population and K is called the carrying capacity, that is the maximum possible size of the population.

   (a) Sketch the direction field associated with this equation.

   (b) State the two equilibrium solutions and label them on the direction field.

   (c) By considering your sketch of the direction field, describe how the population can change with time. [Hints: consider the cases when the initial size of the population is one of the equilibrium solutions and when it is greater or lower than a equilibrium solution. How do the solutions behave near the equilibrium solutions?]

   (d) Solve the differential equation, given that the initial population size is p_0.

   (e) Suppose that $k = \frac{1}{2}$ and that the initial population is 1,000. If after 2 years the population is 500, will the population decrease to zero? If not, find the value around which it stabilizes.

6. The concentration of chlorine in a 600,000 ℓ swimming pool is found to be 0.98 mg/ℓ, which is below the safety threshold of 1 mg/ℓ. In order to ensure that the pool is safe to use, a 2 mg/ℓ mixture of chlorine in water is pumped into the pool at a rate of 20 ℓ per minute and the resulting mixture is pumped out of the pool at the same rate. Assume instantaneous mixing.

   (a) Write down a differential equation that models the quantity of chlorine, q, in the pool at time t.

   (b) Find an expression for q as a function of t.

   (c) How long must it take before the pool is safe to use?

   (d) Assuming that the system continues as above, find the maximum possible quantity of chlorine that can be in the pool at any time.

   (e) The maximum permitted concentration is 3 mg of chlorine per litre of water. Is there any danger of exceeding this maximum concentration?