MA11210 and MT11210 - Differential Equations

2022

Q1

	Order	Degree	Linear?
a)	1	1	✓
b)	2	1	✓
c)	3	2	✗
d)	4	3	✗
e)	5	1	✓

Q2 Integrate w.r.t. x:

$$y = \frac{1}{2}\sin(2x) + 2x^3 + x + A.$$

The boundary condition gives that $A = 7$,

hence $\boxed{y = \frac{1}{2}\sin(2x) + 2x^3 + x + 7}$

Q3 Multiply by an integrating factor $e^{\int(-3x^{-1})dx} = e^{-3\ln x} = x^{-3}$,

to obtain $x^{-3}\frac{dy}{dx} - 3x^{-4}y = \cos 3x + 4e^{-x}$

$$\Leftrightarrow \quad \frac{d}{dx}\left\{x^{-3}y\right\} = \cos 3x + 4e^{-x}$$

$$\Rightarrow \quad x^{-3}y = \frac{1}{3}\sin 3x - 4e^{-x} + A \quad (A \text{ arb. const.})$$

$$\Rightarrow \quad \boxed{y = x^3\left(\frac{1}{3}\sin 3x - 4e^{-x} + A\right)}$$

Q4 This is separable. Thus $\int \frac{dy}{y(y+1)} = \int \frac{dx}{x} + A_1$ (A_1 arb. const.)

We make use of partial fractions: $\int\left(\frac{1}{y} - \frac{1}{1+y}\right)dy = \ln x + A_1$

$$\Rightarrow \ln y - \ln(1+y) = \ln x + \ln A \quad (A \text{ arb. const.})$$

$$\Rightarrow \ln\left(\frac{y}{1+y}\right) = \ln(Ax)$$

$$\Rightarrow \frac{y}{1+y} = Ax$$

$$\Rightarrow \boxed{y = \frac{Ax}{1-Ax}.}$$

$\boxed{\text{Q5}}$ Let $F(x,y) = \dfrac{y^2-x^2}{xy}$. Then for $k \neq 0$, $F(kx,ky) = \dfrac{k^2y^2-k^2x^2}{k^2xy} = F(x,y)$.

The ODE $\frac{dy}{dx} = F(x,y)$ is therefore homogeneous.

Let $v = \frac{y}{x} \iff y = xv$, so $\frac{dy}{dx} = \frac{d}{dx}(xv) = x\frac{dv}{dx} + v$, giving

$$x\frac{dv}{dx} + v = \frac{v^2-1}{v}$$

$$\Rightarrow \frac{dv}{dx} = -\left(\frac{1}{x}\right)\left(\frac{1}{v}\right),$$ which is separable.

Thus $\int v\,dv = -\int \frac{dx}{x} + A \Rightarrow \frac{v^2}{2} = -\ln x + A \Rightarrow \frac{y^2}{2x^2} = A - \ln x$

$$\Rightarrow \boxed{y^2 = 2x^2(A-\ln x).}$$

$\boxed{\text{Q6}}$ Auxiliary equation: $m^2+6m-7=0 \iff (m+7)(m-1)=0 \Rightarrow m=-7$ or $m=1$.

Thus $y_c = Ae^{-7x} + Be^{x}$.

Seek y_p in the form $y_p = ax^2+bx+c$ (so $y_p' = 2ax+b$, $y_p'' = 2a$).

Then $2a + 12ax + 6b - 7ax^2 - 7bx - 7c =$

$$\iff -7ax^2 + (12a-7b)x + (2a+6b-7c) = 21 - 2x - 7x^2$$

Compare coefficients of
x^2: $-7a = -7 \Rightarrow a = 1$
x^1: $12 - 7b = -2 \Rightarrow b = 2$
x^0: $14 - 7c = 21 \Rightarrow c = -1$.

Thus $y_p = x^2+2x-1$ and $\boxed{y = y_c + y_p = Ae^{-7x} + Be^{x} + x^2 + 2x - 1.}$

$\boxed{\text{Q7}}$ Let $z = y^{-1} \iff y = z^{-1}$. Then $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = -z^{-2}\frac{dz}{dx}$. The ODE becomes:

$$-xz^{-2}\frac{dz}{dx} + z^{-1} = x^2z^{-2}\ln x$$

Multiply by $-x^{-1}z^2$

$$\Rightarrow \frac{dz}{dx} - x^{-1}z = -x\ln x,$$ which is first order, linear.

Multiply by an integrating factor $e^{\int -x^{-1}dx} = e^{-\ln x} = x^{-1}$, giving

$$x^{-1}\frac{dz}{dx} - x^{-2}z = -\ln x$$

$$\iff \frac{d}{dx}\left\{x^{-1}z\right\} = -\ln x \Rightarrow x^{-1}z = x(1-\ln x) + A \quad (A \text{ arb. const.})$$

$$\Rightarrow y^{-1} = x^2(1-\ln x) + Ax$$

$$\Rightarrow y = \frac{1}{x^2(1-\ln x) + Ax},$$

and $y(1) = 1 \Rightarrow A = 0$, hence $\boxed{y = x^{-2}(1-\ln x)^{-1}.}$

Q8 Auxiliary equation : $m^2 + 1 = 0 \Rightarrow m = \pm i \Rightarrow x_c = A\cos t + B\sin t$.
Seek x_p in the form $x_p(t) = ae^t = x_p'(t) = x_p''(t)$. Substitute into ODE:
$2ae^t = e^t \Rightarrow a = \frac{1}{2}$; thus $x = x_c + x_p = A\cos t + B\sin t + \frac{1}{2}e^t$.
$x(0) = 1 \Rightarrow A + \frac{1}{2} = 1 \Rightarrow A = \frac{1}{2}$.
$\dot{x}(0) = 1 \Rightarrow B + \frac{1}{2} = 1 \Rightarrow B = \frac{1}{2}$.
Thus $\boxed{x = \frac{1}{2}(\cos t + \sin t + e^t).}$

(8× 5 marks)

SECTION A
40

SECTION B

Q9 a) Auxiliary equation: $m^2 + 2m - 24 = 0 \Leftrightarrow (m-4)(m+6) = 0$, so $m = -6$ or $m = 4$. Thus $y_c = Ae^{-6x} + Be^{4x}$. 1

Seek y_p in the form $y_p = a + bx + cxe^{4x}$ 3

$$\Rightarrow y_p' = b + ce^{4x} + 4cxe^{4x}$$

$$\Rightarrow y_p'' = 8ce^{4x} + 16cxe^{4x}.$$

Substitute into ODE:

$$\cancel{16cxe^{4x}} + 8ce^{4x} + 2b + 2ce^{4x} + \cancel{8cxe^{4x}} - 24a - 24bx - \cancel{24cxe^{4x}} =$$

Compare coefficients of
x^0: $2b - 24a = 8$
x^1: $-24b = 48$
e^{4x}: $10c = 20$

implying $c = 2$, $b = -2$, $a = -\frac{1}{2}$.

Thus $\boxed{y = y_c + y_p = Ae^{-6x} + Be^{4x} + 2xe^{4x} - 2x - \frac{1}{2}.}$ 4

(Continues overleaf)

Q9 cont. b) Auxiliary equation : $m^2 - 2m + 10 = 0 \implies m = \frac{2 \pm \sqrt{4-40}}{2}$

$$= 1 \pm 3i .$$

Thus $y_c = e^x\left(A\cos(3x) + B\sin(3x)\right)$ 1

note $\frac{d}{dx}\left(xe^x\right) = (x+1)e^x$

Seek y_p in the form $y_p = a + xe^x\left(b\cos(3x) + c\sin(3x)\right)$, 3

$$= a + bxe^x\cos(3x) + cxe^x\sin(3x)$$

implying $y_p' = b(x+1)e^x\cos(3x) - 3bxe^x\sin(3x) + c(x+1)e^x\sin(3x) + 3cxe^x\cos(3x)$

$$= (b+3c)xe^x\cos(3x) + (c-3b)xe^x\sin(3x) + be^x\cos(3x) + ce^x\sin(3x)$$

and

$$y_p'' = (b+3c)\left[(x+1)e^x\cos(3x) - 3xe^x\sin(3x)\right] + (c-3b)\left[(x+1)e^x\sin(3x) + 3xe^x\cos(3x)\right]$$

$$+ b\left[e^x\cos(3x) - 3e^x\sin(3x)\right] + c\left[e^x\sin(3x) + 3e^x\cos(3x)\right]$$

$$= xe^x\cos(3x)\left[b + 3c + 3c - 9b\right] + xe^x\sin(3x)\left[-3b - 9c + c - 3b\right]$$

$$+ e^x\cos(3x)\left[b + 3c + b + 3c\right] + e^x\sin(3x)\left[c - 3b - 3b + c\right]$$

$$= (6c - 8b)xe^x\cos(3x) + (-6b - 8c)xe^x\sin(3x) + (2b + 6c)e^x\cos(3x) + (2c - 6b)e^x\sin(3x)$$

Substitute into the ODE:

$$xe^x\cos(3x)\left[\overset{0}{\cancel{6c - 8b - 2b - 6c + 10b}}\right] + xe^x\sin(3x)\left[\overset{0}{\cancel{-6b - 8c - 2c + 6b + 10c}}\right]$$

$$+ e^x\cos(3x)\left[\cancel{2b} + 6c - \cancel{2b}\right] + e^x\sin(3x)\left[\cancel{2c} - 6b - \cancel{2c}\right] + 10a = 1 + e^x\sin(3x)$$

Compare coefficients :

$$\left.\begin{aligned} e^x\cos(3x) &: \quad 6c = 0 \\ e^x\sin(3x) &: \quad -6b = 1 \\ x^0 &: \quad 10a = 1 \end{aligned}\right\} \implies a = \tfrac{1}{10},\ b = -\tfrac{1}{6},\ c = 0.$$

Thus $y_p = \frac{1}{10} - \frac{1}{6}xe^x\cos(3x)$, and so

$$\boxed{y = y_c + y_p = e^x\left(A\cos(3x) + B\sin(3x)\right) - \tfrac{1}{6}xe^x\cos(3x) + \tfrac{1}{10}.}$$

8

20

Q9

$\boxed{\text{Q10}}$ a) Let $u = \frac{x^2 y}{\sin x}$.

Then $\frac{du}{dx} = \frac{2x\sin x - x^2\cos x}{\sin^2 x} y + \frac{x^2}{\sin x}\frac{dy}{dx}$

$= \operatorname{cosec}(x)\left[(2x - x^2\cot x)y + x^2\frac{dy}{dx}\right]$ 3

The ODE becomes $\sin(x)\frac{du}{dx} = \sin^2 x \Rightarrow \frac{du}{dx} = \sin x$

$\Rightarrow u = A - \cos x$

$\Rightarrow x^2 y = A\sin x - \cos x \sin x$

$\Rightarrow y = (A\sin x - \cos x\sin x)/x^2$.

3

Finally, $y(\pi/2) = 1$ implies $1 = \frac{4A}{\pi^2} \Rightarrow A = \frac{\pi^2}{4}$, so

$$\boxed{y = \left(\frac{\pi^2}{4}\sin x - \cos x \sin x\right)/x^2.}$$ 2

b) Note that $\frac{dv}{du} = \frac{dv}{dy}\frac{dy}{dx}\frac{dx}{du} = 1\cdot\frac{dy}{dx}\cdot 1.$ 2

Also, $\frac{y^2 + 2xy - 8y - 10x + 15}{x^2 + 2x + 1} = \frac{v^2 + 2(xy + y - 5x - 5)}{u^2} = \frac{v^2 + 2uv}{u^2}.$ 3

The ODE reduces to $\frac{dv}{du} = \frac{v^2 + 2uv}{u^2}$, which is first order homogeneous.

Let $w = \frac{v}{u} \Leftrightarrow v = uw$, giving $w + u\frac{dw}{du} = \frac{\cancel{u^2}w^2 + 2\cancel{u^2}w}{\cancel{u^2}} \Rightarrow \frac{dw}{du} = \frac{1}{u}(w^2 + w).$

2

This is the same as Q4, giving $w = \frac{Au}{1 - Au}$. 3

In original variables: $\frac{y-5}{x+1} = \frac{A(x+1)}{1 - A(x+1)} \Rightarrow y - 5 = \frac{A(x+1)^2}{1 - A(x+1)}$

$$\Rightarrow \boxed{y = 5 + \frac{A(x+1)^2}{1 - A(x+1)}.}$$ 2

20

Q10

$\boxed{\text{Q11}}$ Auxiliary equation: $m^2 - 2m + 1 = 0 \iff (m-1)^2 = 0 \Rightarrow m = 1$ (repeated).

Thus $y_c = (A + Bx)e^x$ (A, B arb. consts.) 2

Suppose $y_p = c_1(x)e^x + c_2(x)xe^x$ is a particular integral. 3

Then $y_p'(x) = c_1'(x)e^x + c_1(x)e^x + c_2'(x)xe^x + c_2(x)(x+1)e^x$.

Assume $c_1'(x)e^x + c_2'(x)xe^x = 0$. $\circledast$ 2

Then $y_p'(x) = c_1(x)e^x + c_2(x)(x+1)e^x$,

implying $y_p''(x) = c_1'(x)e^x + c_1(x)e^x + c_2'(x)(x+1)e^x + c_2(x)(x+2)e^x$.

Substitute into the ODE:

$$c_1'(x)e^x + \cancel{c_1(x)e^x} + c_2'(x)(x+1)e^x + c_2(x)(\cancel{x}+\cancel{2})e^x \cancel{-2c_1(x)e^x} - 2c_2(x)(\cancel{x}+\cancel{1})e^x + \cancel{c_1(x)e^x} + \cancel{c_2(x)xe^x} = e^x \ln x.$$

$$\Rightarrow \quad c_1'(x)e^x + c_2'(x)(x+1)e^x = e^x \ln x \qquad (\dagger)$$ 5

Solving $\circledast$ and $(\dagger)$ simultaneously, $e^x\begin{pmatrix} 1 & x \\ 1 & x+1 \end{pmatrix}\begin{pmatrix} c_1'(x) \\ c_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ e^x \ln x \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} c_1'(x) \\ c_2'(x) \end{pmatrix} = \begin{pmatrix} x+1 & -x \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \ln x \end{pmatrix} = \begin{pmatrix} -x\ln x \\ \ln x \end{pmatrix}.$$

Now, $\int \ln x \, dx = \int \underset{\substack{u' \\ u = x}}{1} \cdot \underset{\substack{v \\ v' = x^{-1}}}{\ln x} \, dx = x\ln x - x = x(\ln x - 1) = c_2(x)$

and $\int (\underbrace{-x}_{\substack{u' \\ u = -\frac{x^2}{2}}} \underbrace{\ln x}_{\substack{v \\ v' = x^{-1}}}) dx = -\frac{x^2}{2}\ln x + \frac{x^2}{4} = \frac{x^2}{4}(1 - 2\ln x) = c_1(x)$.

Therefore $y_p(x) = \frac{x^2}{4}(1 - 2\ln x)e^x + x^2(\ln x - 1)e^x$, so finally,

$$\boxed{y = y_c + y_p = (A + Bx)e^x + \frac{x^2}{4}(1 - 2\ln x)e^x + x^2(\ln x - 1)e^x.}$$

$$\left(= (A+Bx)e^x - \frac{3x^2e^x}{4} + \frac{x^2e^x\ln x}{2} \right)$$

8

(20) Q11

Q12 a) $\boxed{\begin{array}{l} \frac{dT}{dt} = k(T-10), \\ T(0) = 210. \end{array}}$ 4

b) The ODE is first order and linear. Multiply by an integrating factor e^{-kt} to obtain $e^{-kt}\frac{dT}{dt} - ke^{-kt}T = -10ke^{-kt}$

$$\Leftrightarrow \frac{d}{dt}\left(e^{-kt}T\right) = -10ke^{-kt} \quad 3$$

$$\Rightarrow e^{-kt}T = 10e^{-kt} + A \quad (A \text{ arb. const.})$$

$$\Rightarrow T = 10 + Ae^{kt}. \quad 3$$

The initial condition yields $T(0) = 10 + A = 210 \Rightarrow A = 200$, hence

$$\boxed{T(t) = 10 + 200e^{kt}}$$ is the particular solution.

2

c) $T(1) = 130 \Rightarrow 10 + 200e^{k} = 130 \Rightarrow e^{k} = \frac{120}{200} = \frac{3}{5} \Rightarrow k = \ln\frac{3}{5}$
$= \ln 3 - \ln 5.$

3

Now k is known, we seek t_* such that $T(t_*) = 60$:

$$10 + 200e^{kt_*} = 60 \Rightarrow e^{kt_*} = \frac{50}{200} = \frac{1}{4} \Rightarrow t_* = -\frac{1}{k}\ln 4.$$

Thus $\boxed{t_* = \frac{\ln 4}{\ln 5 - \ln 3} = \frac{2\ln 2}{\ln 5 - \ln 3} \approx 2 \times 1{\cdot}36 = 2{\cdot}72 \text{ minutes.}}$ 5

20 Q12

$\boxed{\text{Q13}}$ a)

$$\boxed{\begin{aligned}\frac{dq}{dt} &= (2)(30) - 30\left(\frac{q}{600,000}\right)\\ &= 60 - \frac{1}{20,000}q\end{aligned}}$$

4

b) This is first order, linear. Multiply by integrating factor $e^{\frac{t}{20,000}}$, giving

$$e^{\frac{t}{20,000}}\frac{dq}{dt} + \frac{e^{\frac{t}{20,000}}}{20,000}q = 60e^{\frac{t}{20,000}}$$

$$\iff \frac{d}{dt}\left(e^{\frac{t}{20,000}}q\right) = 60e^{\frac{t}{20,000}}$$ 3

$$\Rightarrow e^{\frac{t}{20,000}}q = 1,200,000e^{\frac{t}{20,000}} + A \quad (A \text{ arb. const.})$$

$$\Rightarrow q = 1,200,000 + Ae^{-\frac{t}{20,000}}.$$ 3

Since $q(0) = 582,000$, $582,000 = 1,200,000 + A$

$\Rightarrow A = -618,000$,

so $$\boxed{q = 1,200,000 - 618,000\exp(-t/20,000).}$$ 2

c) Seek t_* so that $q = 600,000$:

$$600,000 = 1,200,000 - 618,000e^{-\frac{t_*}{20,000}}$$

$$\Rightarrow 618,000e^{-\frac{t_*}{20,000}} = 600,000$$

$$\Rightarrow e^{-t_*/20,000} = \frac{600}{618}$$

$$\Rightarrow \boxed{t_* = 20,000(\ln 618 - \ln 600) \text{ minutes.}}$$ 5

d) Noting that q is monotone increasing, $\boxed{\lim_{t\to\infty} q(t) = 1,200,000\text{mg.}}$

3

$\textcircled{20}$ Q13