PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI – MEHEFIN / MAY – JUNE 2025

MA11210 – Differential Equations

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Rhoddir marciau llawn am atebion cyflawn i bob cwestiwn yn Rhan A ac i dri cwestiwn yn Rhan B.
- Yn Rhan B, rhoddir credyd am y tri ateb gorau.
- Ni cheir defnyddio cyfrifianellau.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- Full marks will be given for complete answers to all questions in Section A and to three questions in Section B.
- In Section B, credit will be given for the best three answers.
- Calculators are not permitted.
- Students may submit answers to this paper in either Welsh or English.

Section A

1. Write down the order and degree of the following differential equations and state whether or not they are linear:

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \cos x;$

(b) $\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 3y = \cos x;$

(c) $\dfrac{\mathrm{d}^3y}{\mathrm{d}t^3} + t^3\dfrac{\mathrm{d}y}{\mathrm{d}t} - \sin t = 0;$

(d) $\left(\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^2 + \dfrac{\mathrm{d}y}{\mathrm{d}x} = y^2\sin x;$

(e) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + 3xy = \dfrac{x^4\sin y}{1+x^2}.$

[5 marks]

2. Find the solution of the following differential equation that satisfies $y(0) = 2$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 + \cos x.$$

[5 marks]

3. Solve the following first order linear differential equation given that $y(2) = 5$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{3}{x}y = 5x.$$

[5 marks]

4. Find an explicit particular solution of the following differential equation satisfying the condition $y(0) = 0$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2xe^{-y}.$$

[5 marks]

5. Show that the following differential equation is homogeneous and find its general solution:

$$4xy^2\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 + y^3.$$

[5 marks]

6. Find the general solution of the following second order differential equation:

$$y'' - 6y' + 9y = 27x.$$

[5 marks]

7. Find the general solution of the following Bernoulli equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4y - 4xy^3.$$

[5 marks]

8. Find the solution of the following differential equation that satisfies the specified initial conditions:

$$\ddot{x} + 9x = 10e^t; \quad x(0) = 1, \quad \dot{x}(0) = 0.$$

[5 marks]

Section B

9. Find the general solutions of the following differential equations:

(a) $y'' - 2y' - 3y = 18e^{2x} + 6x^2 + 8x + 2$; [10 marks]

(b) $y'' - 4y' + 4y = 6e^{2x} + 8$. [10 marks]

10. Clearly explaining your working, use Lagrange's method of variation of parameters to find the general solution of the following differential equation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = \frac{e^x}{x^2+1}.$$

[20 marks]

11. Find the implicit general solution of the following differential equations:

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x+y+2}{x-y+4}$; (b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x+y+2}{x+y+4}$. [12,8 marks]

12. In the *"Wheely Good Pets"* pet shop (open 7 days a week for all your pet-based needs), a colony of hamsters grows at a rate proportional to its current population. In the absence of any interference, the colony population size doubles every week. Initially, there are 300 hamsters. However, due to their *"Squeak Squad: Tiny Paws, Huge Hearts!"* marketing campaign, 20 hamsters are adopted each day.

(a) Formulate and solve a differential equation that determines the number of hamsters as a function of time (in weeks). [12 marks]

(b) Describe the long-term behaviour of the hamster population size. You may find the fact that $e^{0.7} \approx 2$ useful. [3 marks]

(c) Derive an expression for the *minimum* number of hamsters that must be adopted per day to ensure that the pet shop eventually runs out of hamsters? [5 marks]

13. *"Kevin's Cambrian Concoctions"*, an Aberystwyth cocktail bar, has a giant punch bowl holding 1000 litres of a fruity punch with an unknown concentration of sugar. In his latest experiment to craft the perfect beverage, Kevin starts pumping in a special mixture at a rate of 8 litres per minute; this incoming liquid contains 100 grams of sugar per litre. To prevent a sticky situation, the well-stirred punch is drained at the same rate, with the run-off liquid being frozen into sweet ice lollies.

(a) Formulate and solve a differential equation that determines the quantity of sugar in the punch bowl at time t. Give a physical interpretation of any arbitrary constants in your solution. [8 marks]

(b) After a long time, how much sugar is in the punch bowl? Does this value depend on the amount present initially? [2 marks]

(c) The run-off mixture, at an initial temperature of $20^\circ\mathrm{C}$ is poured into moulds which are placed in a freezer that maintains a steady ambient temperature of $-20^\circ\mathrm{C}$. After 5 minutes in the freezer, the mixture has temperature $5^\circ\mathrm{C}$. Use Newton's Law of Cooling to determine how much time in the freezer elapses before the mixture's temperature reaches $-10^\circ\mathrm{C}$.

Hint: You may find the fact that $\ln 2/(\ln 8 - \ln 5) \approx 1.47$ *useful.* [10 marks]