MA11210 – Differential Equations : 2023

Q1

	Order	Degree	Linear?
a)	1	1	✓
b)	2	1	✓
c)	3	2	✗
d)	4	3	✗
e)	1	1	✓

5

Q2 Integrate w.r.t. x:

$$y = \tfrac{1}{5}\sin(5x) + 3x^3 + 5x + A \quad (A \text{ arb. const.})$$

The condition $y(0) = 1$ implies $A = 1$, so

$$\boxed{y = \tfrac{1}{5}\sin(5x) + 3x^3 + 5x + 1}$$

5

Q3 Multiply by an integrating factor $e^{-\int 4x^{-1}dx} = e^{-4\ln x} = (e^{\ln x})^{-4} = x^{-4}$, to obtain

$$x^{-4}\frac{dy}{dx} - 4x^{-5} = \cos(2x)$$

$$\Leftrightarrow \frac{d}{dx}\left(x^{-4}y\right) = \cos(2x)$$

$$\Rightarrow x^{-4}y = \tfrac{1}{2}\sin(2x) + c$$

$$\Rightarrow y = \tfrac{1}{2}x^4\sin(2x) + cx^4.$$

The condition $y(\pi) = 1$ implies $1 = c\pi^4$, so $c = \pi^{-4}$.

Thus $\boxed{y = x^4\left(\tfrac{1}{2}\sin(2x) + \pi^{-4}\right)}$ 5

Q4 The equation is separable: $\displaystyle\int \frac{dy}{1+y^2} = \int dx \Rightarrow \arctan y = x + c$

$$\Rightarrow y = \tan(x + c)$$

Applying the condition $y(0) = 1 \Rightarrow 1 = \tan(c) \Rightarrow c = \frac{\pi}{4}$, so $\boxed{y = \tan\left(x + \frac{\pi}{4}\right).}$

5

Q5 Writing $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{y^2}{x^2} = F(x,y)$, note that for $k \neq 0$, $F(kx, ky) = \dfrac{ky}{kx} + \dfrac{k^2y^2}{k^2x^2} = F(x,y)$.

The ODE is thus first order homogeneous.

Let $v = \frac{y}{x} \Leftrightarrow y = xv$. Then $\cancel{v} + x\dfrac{dv}{dx} = \cancel{v} + v^2 \Rightarrow \int v^{-2}\,dv = \int x^{-1}dx$, and so

$-v^{-1} = \ln|x| + c$, where c is an arbitrary constant.

Thus $-\dfrac{x}{y} = \ln|x| + c \Rightarrow y = \dfrac{x}{A - \ln|x|}$.

Applying $y(1) = 1$ gives $1 = \frac{1}{A} \Rightarrow A = 1$, so $\boxed{y = \dfrac{x}{1 - \ln|x|}}$ 5

Q6 Auxiliary equation: $m^2 - 6m - 7 = 0 \Leftrightarrow (m-7)(m+1) = 0 \Rightarrow m = -1$ or $m = 7$.

Thus the complementary function is $y_c = Ae^{-x} + Be^{7x}$.

Seek the particular integral in the form $y = ae^x$. Substituting into the ODE:

$-12ae^x = 12e^x \Rightarrow a = -1$, giving $y_p = -e^x$.

Combining the above, $\boxed{y = y_c + y_p = Ae^{-x} + Be^{7x} - e^x.}$ 5

$\boxed{Q7}$ Auxiliary equation : $m^2 + 2 = 0 \Rightarrow m = \pm i\sqrt{2}$.

Thus the complementary function is $x_c = A\cos(\sqrt{2}t) + B\sin(\sqrt{2}t)$.

Seek the particular integral in the form $x_p(t) = at^2 + bt + c \Rightarrow x_p''(t) = 2a$.

Substitute into the ODE: $2a + 2at^2 + 2bt + 2c = 10t^2$.

Comparing coefficients of
t^2 : $2a = 10 \Rightarrow a = 5$
t : $2b = 0 \Rightarrow b = 0$
t^0 : $2(a+c) = 0 \Rightarrow c = -5$

Thus $x = x_c + x_p = A\cos(\sqrt{2}t) + B\sin(\sqrt{2}t) + 5(t^2 - 1)$.

The condition $x(0) = 1$ implies $A - 5 = 1 \Rightarrow A = 6$.

The condition $\dot{x}(0) = 0$ implies $B\sqrt{2} = 0 \Rightarrow B = 0$.

The particular solution is therefore $\boxed{x(t) = 6\cos(\sqrt{2}t) + 5(t^2 - 1).}$ 5

$\boxed{Q8}$ Rearranging, $\frac{dy}{dx} - x^{-1}y = x^{-3}y^2$. Let $z = y^{-1}$. Then $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$
$= -y^{-2}\frac{dy}{dx}$

Thus $-z^{-2}\frac{dz}{dx} - x^{-1}z^{-1} = x^{-3}z^{-2}$

$\Rightarrow \frac{dz}{dx} + x^{-1}z = -x^{-3}$, which is first order, linear.

Multiply by an integrating factor x: $x\frac{dz}{dx} + z = -x^{-2}$

$\Leftrightarrow \frac{d}{dx}(xz) = -x^{-2}$

$\Rightarrow xz = x^{-1} + A$ (A arb. const)

$\Rightarrow z = x^{-2} + Ax^{-1}$

$\Rightarrow \boxed{y = \frac{1}{x^{-2} + Ax^{-1}} = \frac{x^2}{1 + Ax}}$. 5

SECTION A
40

SECTION B

Q9 a) We first seek y_c, the complementary function. The auxiliary equation is $m^2 + 5m - 24 = 0 \iff (m-3)(m+8) = 0$, and so $y_c = Ae^{-8x} + Be^{3x}$. 2

Since a term in the complementary function is already of the form ae^{3x}, we seek the particular integral y_p in the form $y_p = axe^{3x}$ 2

$$\Rightarrow y_p' = 3axe^{3x} + ae^{3x}$$

$$\Rightarrow y_p'' = 9axe^{3x} + 3ae^{3x} + 3ae^{3x}$$
$$= 9axe^{3x} + 6ae^{3x}.$$

Substitution into the ODE yields:

$$\cancel{9axe^{3x}} + 6ae^{3x} + \cancel{15axe^{3x}} + 5ae^{3x} - \cancel{24axe^{3x}} = 22e^{3x}$$
$$\iff \quad 11ae^{3x} = 22e^{3x}$$
$$\Rightarrow \quad a = 2,$$

so $y_p = 2xe^{3x}$. Combining the above, $\boxed{y = Ae^{-8x} + Be^{3x} + 2xe^{3x}.}$ 4

b) We first seek y_c, the complementary function. The auxiliary equation is $m^2 - 4m + 5 = 0 \iff m_{\pm} = \frac{4 \pm 2i}{2} = 2 \pm i$. It follows that $y_c = e^{2x}(A\cos x + B\sin x)$. 2

Seek the particular integral in the form $y_p = ae^x\sin x + be^x\cos x + cx + d$. 4

$$\Rightarrow y_p' = ae^x\sin x + ae^x\cos x + be^x\cos x - be^x\sin x + c$$
$$= (a-b)e^x\sin x + (a+b)e^x\cos x + c$$

$$\Rightarrow y_p'' = (a-b)e^x\sin x + (a-b)e^x\cos x + (a+b)e^x\cos x - (a+b)e^x\sin x$$
$$= -2be^x\sin x + 2ae^x\cos x,$$

Substitute into the ODE:

$$e^x\sin x\left[-2b - 4a + 4b + 5a\right] + e^x\cos x\left[2a - 4a - 4b + 5b\right] + 5cx + 5d - 4c = 10e^x\sin x + 125x.$$

Comparing coefficients of

$$\left.\begin{array}{ll} e^x\sin x: & a + 2b = 10 \\ e^x\cos x: & b - 2a = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} a = 2, \\ b = 4, \end{array}\right.$$

$$x: \quad 5c = 125 \quad \Rightarrow \quad c = 25,$$
$$x^0: \quad 5d - 4c = 0 \quad \Rightarrow \quad d = 20.$$

Thus $\boxed{y = y_c + y_p = e^{2x}(A\cos x + B\sin x) + 2e^x\sin x + 4e^x\cos x + 25x + 20.}$ 6

20

Q10 Let $x = e^t \Leftrightarrow t = \ln x$. Then $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt} = e^{-t}\frac{dy}{dt}$ 3

Moreover, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(e^{-t}\frac{dy}{dt}\right) = \frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)\frac{dt}{dx}$

$$= e^{-t}\left(e^{-t}\frac{d^2y}{dt^2} - e^{-t}\frac{dy}{dt}\right)$$

$$= e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$ 4

Substituting these results into the ODE gives:

$$\cancel{e^{2t}}\,\cancel{e^{-2t}}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - 3\cancel{e^{t}}\,\cancel{e^{-t}}\frac{dy}{dt} + 4y = 4t$$

$$\Leftrightarrow \quad \frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 4t,$$ 3

which is second order, linear, and has constant coefficients.

Auxiliary equation: $(m-2)^2 = 0 \Rightarrow m = 2$ (repeated), so $y_c = (A+Bt)e^{2t}$. 2

Seek y_p in the form $y_p = a + bt \Rightarrow y_p' = b \Rightarrow y_p'' = 0$.

Substituting into the ODE: $-4b + 4bt + 4a = 4t$.

Compare coefficients of t: $4b = 4 \Rightarrow b = 1$

Compare coefficients of t^0: $4a - 4 = 0 \Rightarrow a = 1$.

Thus $y_p = 1 + t$ and so $y = (A+Bt)e^{2t} + 1 + t$.

$$= (A + B\ln x)x^2 + 1 + \ln x.$$ 3

The condition $y(1) = 0$ implies $A + 1 = 0 \Rightarrow A = -1$.

The condition $y'(1) = 1$ implies $\left.\left(-2x + Bx + 2Bx\ln x + x^{-1}\right)\right|_{x=1} = 0$

$$\Rightarrow \quad -2 + B + 1 = 1$$

$$\Rightarrow \quad B = 2.$$

Combining the above, $\boxed{y = (2\ln x - 1)x^2 + \ln x + 1.}$ 5

20

Q11. a) (i) Let $x = u + x_0$, $y = v + y_0$. Then $\frac{dv}{du} = \frac{dv}{dy}\frac{dy}{dx}\frac{dx}{du}$

$= 1 \cdot \frac{dy}{dx} \cdot 1$,

so the ODE becomes $$\frac{dv}{du} = \frac{au + bv + (c + ax_0 + by_0)}{lu + mv + (n + lx_0 + my_0)}.$$ 2

The ODE is first-order homogeneous if the parenthetical terms are set to zero. That is:

$$\begin{pmatrix} a & b \\ l & m \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -c \\ -n \end{pmatrix} \Rightarrow \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \frac{1}{bl - am}\begin{pmatrix} m & -b \\ -l & a \end{pmatrix}\begin{pmatrix} c \\ n \end{pmatrix} = \frac{1}{bl-am}\begin{pmatrix} cm - bn \\ an - cl \end{pmatrix}.$$ 2

Here, we have assumed $bl - am \neq 0$. If this assumption does not hold, such a choice of x_0 and y_0 is not possible. 1

(ii) By the above (with $a = b = c = l = n = 1$ and $m = -1$, noting $bl - am = 2 \neq 0$), let $x_0 = -1$ and $y_0 = 0$. That is, let $x = u - 1$, $y = v$. 2

Then $\frac{dv}{du} = \frac{u+v}{u-v}$, which is first order homogeneous.

Let $w = \frac{v}{u} \Leftrightarrow v = uw$. Then $\frac{dv}{du} = \frac{d}{du}(uw) = u\frac{dw}{du} + w = \frac{1+w}{1-w}$,

and so $$\frac{dw}{du} = \frac{1}{u}\left(\frac{1+w}{1-w} - w\right) = \frac{1}{u}\left(\frac{1 + w - w + w^2}{1-w}\right) = \frac{1}{u}\left(\frac{1+w^2}{1-w}\right),$$

which is clearly separable. 4 Thus $\int \frac{1-w}{1+w^2}\,dw = \int \frac{du}{u} + c$

$$\Rightarrow \int \frac{dw}{1+w^2} - \frac{1}{2}\int \frac{2w}{1+w^2}\,dw = \ln u + c$$

$$\Rightarrow \arctan w - \frac{1}{2}\ln(1 + w^2) = \ln u + c$$

$$\Rightarrow \arctan\left(\frac{y}{x+1}\right) - \frac{1}{2}\ln\left(1 + \left(\frac{y}{x+1}\right)^2\right) = \ln(x+1) + c$$

(other simplifications acceptable) 4

b) $bl - am = 1(-1) - 1(-1) = 0$, contradicting (a)'s assumption. 1

Let $u = x + y$. Then $\frac{dy}{dx} = \frac{d}{dx}(u - x) = \frac{du}{dx} - 1 = \frac{u}{3-u}$

$$\Rightarrow \frac{du}{dx} = \frac{u + 3 - u}{3 - u} = \frac{3}{3-u},$$ 2

which is separable. Thus $\int (3-u)\,du = \int 3\,dx \Rightarrow 3u - \frac{u^2}{2} = 3x + c$

$$\Rightarrow 3(x+y) - \frac{(x+y)^2}{2} = 3x + c$$

$$\Rightarrow 3y - \frac{(x+y)^2}{2} = c,$$ 2

where c is an arbitrary constant (other simplifications acceptable).

20

Q12. a) $\frac{dT}{dt} = k(T - T_A)$

$T(0) = T_0$ 4

b) Multiply by an integrating factor e^{-kt}, giving $e^{-kt}\frac{dT}{dt} - ke^{-kt}T = -kT_A e^{-kt}$

$$\Leftrightarrow \frac{d}{dt}\left(e^{-kt}T\right) = -kT_A e^{-kt}$$

$$\Rightarrow e^{-kt}T = T_A e^{-kt} + A \quad (A \text{ arb. const.})$$

$$\Rightarrow T = T_A + Ae^{kt}.$$ 4

The initial condition gives $T_0 = T_A + A \Rightarrow A = T_0 - T_A$, whence

$$T(t) = T_A + (T_0 - T_A)e^{kt}.$$ 2

k is negative to ensure cooling (as $t \to \infty$, $T \to T_A$). 2

c) The information in this part of the question gives:

$$T_0 = 90$$
$$T_A = 20$$
$$T(30) = 80.$$

The particular solution becomes $T(t) = 20 + 70e^{kt}$. 2

$T(30) = 80$ implies $80 = 20 + 70e^{30k}$

$$\Rightarrow 60 = 70e^{30k}$$
$$\Rightarrow \frac{6}{7} = e^{30k}$$
$$\Rightarrow k = \frac{1}{30}(\ln 6 - \ln 7).$$ 3

We seek t_* such that $T(t_*) = 50$

$$\Rightarrow 20 + 70e^{kt_*} = 50$$
$$\Rightarrow e^{kt_*} = \frac{3}{7}$$
$$\Rightarrow t_* = \frac{1}{k}(\ln 3 - \ln 7) = 30\,\frac{\ln 3 - \ln 7}{\ln 6 - \ln 7}$$
$$\approx 30 \times 5.50$$
$$= 165 \text{ seconds}.$$ 3

20

$\boxed{Q13.}$ a) Let $N(t)$ denote the number of locusts at time t (weeks).

In the absence of other factors, $\frac{dN}{dt} = kN$, which has solution $N(t) = Ae^{kt}$. The initial condition gives $N(0) = A$.

The doubling condition gives $N(1) = 2A$, so $2A = Ae^{k} \Rightarrow k = \ln 2$.

3

Introducing predation,

$$\frac{dN}{dt} = (\ln 2)N - p$$

$$\Leftrightarrow \quad \frac{dN}{dt} - (\ln 2)N = -p,$$

which is first order and linear. Multiply by an integrating factor $e^{-t\ln 2} = 2^{-t} = e^{-kt}$, giving:

$$e^{-kt}\frac{dN}{dt} - ke^{-kt}N = -pe^{-kt}$$

$$\Leftrightarrow \quad \frac{d}{dt}\left(e^{-kt}N\right) = -pe^{-kt}$$

$$\Rightarrow \quad e^{-kt}N = \frac{p}{k}e^{-kt} + A \quad (A \text{ arb. const.})$$

$$\Rightarrow \quad N = \frac{p}{\ln 2} + A \cdot 2^{t}$$

5

The initial condition $N(0) = 500{,}000$ implies $500{,}000 = \frac{p}{\ln 2} + A$

$$\Rightarrow \quad A = 500{,}000 - \frac{p}{\ln 2}.$$

Combining, $N(t) = \frac{p}{\ln 2} + \left(500{,}000 - \frac{p}{\ln 2}\right) \cdot 2^{t}$.

4

b) The population dies out if $500{,}000 - \frac{p}{\ln 2} < 0$

$$\Rightarrow \quad p > 500{,}000(\ln 2).$$

3

If this inequality holds, then the population dies out when

$$N(t_*) = 0 \quad \Rightarrow \quad \frac{p}{\ln 2} + \left(500{,}000 - \frac{p}{\ln 2}\right) \cdot 2^{t_*} = 0$$

$$\Rightarrow \quad e^{(\ln 2)t_*} = \frac{-\left(\frac{p}{\ln 2}\right)}{500{,}000 - \frac{p}{\ln 2}}$$

$$\Rightarrow \quad t_* = \frac{1}{\ln 2}\ln\left(\frac{p}{p - 500{,}000\ln 2}\right).$$

5

20

SECTION B
60