

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI – MEHEFIN / MAY – JUNE 2023

MA11210 - Differential Equations

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

Time allowed - 2 hours

- Rhoddir marciau llawn am atebion cyflawn i bob cwestiwn yn Rhan A ac i dri cwestiwn yn Rhan B.
- Yn Rhan B, rhoddir credyd am y tri ateb gorau.
- Ni cheir defnyddio cyfrifianellau.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.
- Full marks will be given for complete answers to all questions in Section A and to three questions in Section B.
- In Section B, credit will be given for the best three answers.
- Calculators are not permitted.
- Students may submit answers to this paper in either Welsh or English.

Ar ôl eistedd, gall myfyrwyr lenwi tudalen flaen y llyfryn atebion a'r papur presenoldeb.

Once seated, students may complete the front cover of the answer book and the attendance slip.

Peidiwch ag agor y papur arholiad tan y dywedir wrthyfych am wneud hynny.

Do not open the question paper until instructed to do so.

Section A

1. Write down the order and degree of the following differential equations and state whether or not they are linear:

(a) $\frac{dy}{dx} + 6y = x^3;$

(b) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = \sin x;$

(c) $\left(\frac{d^3y}{dx^3}\right)^2 + 5x\left(\frac{dy}{dx}\right)^6 = 7;$

(d) $\frac{d^2y}{dx^2} - \left(\frac{d^4y}{dx^4}\right)^3 = y^2;$

(e) $x^3\frac{dy}{dx} = 1.$

[5 marks]

2. Find the solution of the following differential equation that satisfies $y(0) = 1$:

$$\frac{dy}{dx} = \cos(5x) + 9x^2 + 5.$$

[5 marks]

3. Solve the following linear first order differential equation given that $y(\pi) = 1$:

$$\frac{dy}{dx} - \frac{4}{x}y = x^4 \cos(2x).$$

[5 marks]

4. Find the explicit particular solution of the following differential equation satisfying the condition $y(0) = 1$:

$$\frac{dy}{dx} = 1 + y^2.$$

[5 marks]

5. Show that the following differential equation is homogeneous and find its explicit particular solution that satisfies $y(1) = 1$:

$$x^2\frac{dy}{dx} = xy + y^2.$$

[5 marks]

6. Find the general solution of the following second order differential equation:

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} - 7y = 12e^x.$$

[5 marks]

7. Find the solution of the following differential equation that satisfies the specified initial conditions:

$$\ddot{x} + 2x = 10t^2; \quad x(0) = 1, \quad \dot{x}(0) = 0.$$

[5 marks]

8. Find the explicit general solution of the following Bernoulli equation:

$$x^3\frac{dy}{dx} = y(x^2 + y).$$

[5 marks]

Section B

9. Find the general solutions of the following differential equations:

(a) $y'' + 5y' - 24y = 22e^{3x}$; [8 marks]

(b) $y'' - 4y' + 5y = 10e^x \sin(x) + 125x$. [12 marks]

10. Using the substitution $x = e^t$ or otherwise, find the solution of the equation

$$x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = 4 \ln x, \quad x > 0,$$

which satisfies the conditions $y(1) = 0$ and $y'(1) = 1$. [20 marks]

11. (a) (i) Consider differential equations of the form:

$$\frac{dy}{dx} = \frac{ax + by + c}{lx + my + n},$$

where a, b, c, l, m, n are given constants. Find constants x_0 and y_0 so that the change of variables $x = u + x_0$ and $y = v + y_0$ reduce the differential equation to one of homogeneous type. State clearly any assumptions you have made about the values of a, b, c, l, m, n . [5 marks]

(ii) Hence or otherwise, find the general solution of the following differential equation:

$$\frac{dy}{dx} = \frac{x + y + 1}{x - y + 1}.$$

[10 marks]

(b) Consider the differential equation

$$\frac{dy}{dx} = \frac{x + y}{3 - x - y}.$$

Briefly explain why the method described in (a) is not suitable for solving this equation. Using the change of variable $u = x + y$ or otherwise, find its general solution. [5 marks]

12. Newton's law of cooling states that the rate of change of temperature in a cooling body is directly proportional to the difference in temperature between the body and its surroundings.

Sue Fflay, the head chef at an Aberystwyth restaurant, prepares soup for a large end-of-term party of mathematics students. At the instant the soup leaves the kitchen, it has an initial temperature T_0 , while the restaurant has ambient temperature T_A .

- (a) Write down a differential equation for the temperature T (a function of time t) of the soup, along with an initial condition. [4 marks]
- (b) Use an appropriate method to find the explicit particular solution of this differential equation, leaving only one unknown constant (consider T_0 and T_A to be known) in your solution. Would you expect the unknown constant to be positive or negative? Why? [8 marks]
- (c) Suppose the soup had initial temperature 90°C , while the restaurant's temperature is 20°C . From her extensive past experience, the chef knows that in 30 seconds, soup cools to 80°C .

Immediately upon leaving the kitchen, the waitress gets distracted by a particularly extroverted mathematician who starts a long and fascinating conversation with her about differential equations. Determine how long a conversation she can have (to the nearest second) before the soup reaches an unsatisfyingly tepid serving temperature of 50°C . [8 marks]

[You may find the fact that $\frac{\ln 7 - \ln 3}{\ln 7 - \ln 6} \approx 5.50$ useful.]

13. The population of locusts in a certain area increases at a rate proportional to the current population size. In the absence of other factors (such as predation) the population would double each week. Initially, the area contains 500,000 locusts. Predators such as birds and rodents eat a constant number, p say, of locusts per week.

- (a) Determine the population of locusts at any time. [12 marks]
- (b) What is the minimum value of p for which the population of locusts eventually dies out? Providing p exceeds this value, state the time taken for the population to die out in terms of p . [8 marks]

END OF PAPER