ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI / MAY 2022

MA11210 - Differential Equations

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Rhoddir marciau llawn am atebion cyflawn i bob cwestiwn yn Rhan A ac i dri cwestiwn yn Rhan B.
- Yn Rhan B, rhoddir credyd am y tri ateb gorau.
- Ni cheir defnyddio cyfrifianellau.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- Full marks will be given for complete answers to all questions in Section A and to three questions in Section B.
- In Section B, credit will be given for the best three answers.
- Calculators are not permitted.
- Students may submit answers to this paper in either Welsh or English.

Section A

1. Write down the order and degree of the following differential equations and state whether or not they are linear:

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + 6y = x^3;$

(b) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sin x;$

(c) $\left(\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)^2 + 5x\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^6 = 7;$

(d) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - \left(\dfrac{\mathrm{d}^4y}{\mathrm{d}x^4}\right)^3 = y^2;$

(e) $\sin x\dfrac{\mathrm{d}y}{\mathrm{d}x} - e^x\dfrac{\mathrm{d}^5y}{\mathrm{d}x^5} = 0.$

[5 marks]

2. Find the solution of the following differential equation that satisfies $y(0) = 7$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos(2x) + 6x^2 + 1.$$

[5 marks]

3. Find the explicit general solution of the following linear first order differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{3}{x}y = x^3\cos 3x + 4x^3e^{-x}.$$

[5 marks]

4. Find the explicit general solution of the following differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(y+1)}{x}.$$

[5 marks]

5. Show that the following differential equation is homogeneous and find its general solution:

$$xy\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - x^2.$$

[5 marks]

6. Find the general solution of the following second order differential equation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} - 7y = 21 - 2x - 7x^2.$$

[5 marks]

7. Find the explicit particular solution of the following Bernoulli equation which satisfies the condition $y(1) = 1$:

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = y^2x^2\ln x.$$

Hint: $\ln x$ *can be integrated by parts.* [5 marks]

8. Find the solution of the following differential equation that satisifies the specified initial conditions:

$$\ddot{x} + x = e^t; \quad x(0) = 1, \quad \dot{x}(0) = 1.$$

[5 marks]

Section B

9. Find the general solutions of the following differential equations:

(a) $y'' + 2y' - 24y = 8 + 48x + 20e^{4x}$; [8 marks]

(b) $y'' - 2y' + 10y = 1 + e^{x}\sin(3x)$. [12 marks]

10. (a) By using the substitution $u = x^2 \operatorname{cosec}(x) y$, or otherwise, find the explicit particular solution of the following ODE that satisfies $y(\pi/2) = 1$:

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = \sin^2 x + (x^2 \cot x - 2x)y.$$

[8 marks]

(b) Using the substitutions $v = y - 5$ and $u = x + 1$, find the explicit general solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 + 2xy - 8y - 10x + 15}{x^2 + 2x + 1}.$$

[12 marks]

11. Clearly explaining your working, use Lagrange's method of variation of parameters to find the general solution of

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = e^{x}\ln x.$$

[20 marks]

12. Newton's law of cooling states that the rate of change of temperature in a cooling body is directly proportional to the difference in temperature between the body and its surroundings.

At a beach bonfire after sunset on a cool May evening (temperature 10°C), a banana with chocolate and marshmallows is wrapped in tin foil and placed in the fire, with the intention of baking the banana and melting the chocolate and marshmallows.

At the moment it is lifted from the fire, the banana package has temperature 210°C.

(a) Write down a differential equation for the temperature T (a function of time t) of the banana, along with an initial condition. [4 marks]

(b) Use an appropriate method to find the explicit particular solution of this differential equation, which will still contain one unknown constant. [8 marks]

(c) One minute after removal from the barbecue, the banana has cooled to 130°C. Determine in minutes (to two decimal places) how long after lifting out of fire the banana will have cooled to a comfortably eatable temperature of 60°C. [8 marks]
[*You may find the fact that* $\frac{\ln 2}{\ln 5 - \ln 3} \approx 1.36$ *useful*]

13. The concentration of chlorine in a 600,000 ℓ swimming pool is found to be 0.97 mg/ℓ, which is below the safety threshold of 1 mg/ℓ. In order to ensure that the pool is safe to use, a 2 mg/ℓ mixture of chlorine in water is pumped into the pool at a rate of 30 ℓ per minute and the resulting mixture is pumped out of the pool at the same rate. Assume instantaneous mixing.

(a) Write down a differential equation that models the quantity of chlorine, q, in the pool at time t. [4 marks]

(b) Find an expression for q as a function of t. [8 marks]

(c) How long must it take before the pool is safe to use?
(You may leave your answer in exact form; there is no need to numerically evaluate your answer.) [5 marks]

(d) Assuming that the system continues as above, find the maximum possible quantity of chlorine that can be in the pool at any time. [3 marks]